A cylindrically symmetric magnetic field in the presence of freely moving particles

1968 J. Phys. A: Gen. Phys. 1 108

(http://iopscience.iop.org/0022-3689/1/1/314)

J. PHYS. A (PROC. PHYS. SOC.), 1968, SER. 2, VOL. 1. PRINTED IN GREAT BRITAIN

A cylindrically symmetric magnetic field in the presence of freely moving particles

M. M. SOM

Physics Department, City College, Calcutta, India

MS. received 20*th June* 1967

Abstract. The paper demonstrates the existence of a solution of the Einstein–Maxwell equations in general relativity where matter coexists with a cylindrically symmetric axial field.

1. Introduction

In two previous papers the author (Som 1964, 1967) has shown that Einstein–Maxwell's equations do not permit an acceptable solution corresponding to a cylindrically symmetric radial electrostatic (or magnetostatic) field. It seems worth while to investigate whether a cylindrically symmetric axial field is consistent with the Einstein–Maxwell equations in general relativity. In the literature there already exists such a singularity-free solution—the so-called Melvin magnetic universe (Melvin 1964). However, the Melvin universe is a purely magnetic universe having no matter anywhere. In the present paper explicit solutions are obtained where matter coexists with an axial magnetic field. Matter here is presented in the form of two clusters of particles moving in circles in counter directions, as in a previous paper by Raychoudhuri and Som (1962), so that in the absence of the magnetic field the solutions reduce to those previously found. As the particles are uncharged, the interaction is purely gravitational.

2. Basic equation

For regions in which there are both matter and magnetic field, the Einstein–Maxwell equations are

$$R^{\mu}_{\nu}-\tfrac{1}{2}\delta^{\mu}_{\nu}R = -8\pi(T^{\mu}_{\nu}+E^{\mu}_{\nu}) \tag{1}$$

with

$$E^{\mu}_{\nu} = -F^{\mu\alpha}F_{\nu\alpha}+\tfrac{1}{4}\delta^{\mu}_{\nu}F^{\alpha\beta}F_{\alpha\beta} \tag{2}$$

and

$$T^{\mu}_{\nu} = \sum_{i} \rho_i v^{\mu}_{i} v_{i\nu} \tag{3}$$

where ρ_i is the density of the ith group of particles having velocity vector $v_i{}^{\mu}$. In the present case there are only two groups, so that $i = 1, 2$ and

$$F^{\mu\nu}{}_{;\nu} = 0 \tag{4a}$$

$$F_{[\mu\nu;\alpha]} = 0. \tag{4b}$$

We number the radial, axial and the angular coordinates r, z, ϕ as 1, 2, 3, so that for a purely axial magnetic field the only non-vanishing component of $F_{\alpha\beta}$ is F_{13}. Equations (4b) are then automatically satisfied as F_{13} is a function of r alone. Therefore $E^1_1 = -E^2_2$ and $T^1_1 = T^2_2 = 0$, so that we obtain

$$R^3_3+R^4_4 = 0.$$

The cylindrically symmetric line element may therefore be taken in Weyl's canonical form (Synge 1960)

$$ds^2 = \exp(2\alpha)\,dt^2-\exp(2\beta-2\alpha)(dr^2+dz^2)-r^2\exp(-2\alpha)\,d\phi^2 \tag{5}$$

where α and β are functions of r alone owing to the assumed symmetry.

Since there is no non-gravitational interaction, the particles may be assumed to follow geodesics as in a paper by Einstein (1939), and hence

$$\frac{\partial \nu^\alpha}{\partial s} + \Gamma^\alpha_{\mu\nu} \nu^\mu \nu^\nu = 0. \tag{6}$$

For a purely circular motion in the (r, ϕ) plane $dr/ds = dz/ds = 0$. Thus, with the line element (5), equation (6) reduces to

$$\Gamma^1_{33}(\nu^3)^2 + \Gamma^1_{44}(\nu^4)^2 = 0 \tag{7}$$

$$\frac{d\nu^3}{ds} = \frac{d\nu^2}{ds} = 0. \tag{8}$$

From equations (5) and (7), we obtain

$$(\nu^3)^2 = \frac{e^{2\alpha}\alpha'}{r(1-2r\alpha')} \tag{9}$$

$$(\nu^4)^2 = \frac{e^{-2\alpha}(1-r\alpha')}{1-2r\alpha'}. \tag{10}$$

Equations (9) and (10) demand that our real system must satisfy the condition

$$r\alpha' < \tfrac{1}{2}. \tag{11}$$

Now, from equation (4) we have

$$F_{31} = kr\, e^{-2\alpha} \tag{12}$$

k being a constant. The surviving components of E^μ are then

$$E^1_1 = E^3_3 = -E^2_2 = -E^4_4 = -\tfrac{1}{2}k^2\, e^{-2\beta} \tag{13}$$

and the surviving components of T^μ_ν are

$$T^3_3 = \rho\nu^3\nu_3 = -\frac{\rho r\alpha'}{1-2r\alpha'} \tag{14}$$

$$T^4_4 = \rho\nu^4\nu_4 = \frac{\rho(1-r\alpha')}{1-2r\alpha'}. \tag{15}$$

T^4_3 vanishes owing to the assumption that equal numbers of particles are moving in counter directions.

The field equations now give

$$\frac{\beta'}{r} - \alpha'^2 = 4\pi k^2\, e^{-2\alpha} \tag{16}$$

$$\exp(2\alpha-2\beta)(\beta''+\alpha'^2) = 8\pi\rho\frac{r\alpha'}{1-2r\alpha'} + 4\pi k^2 \exp(-2\beta) \tag{17}$$

$$\exp(2\alpha-2\beta)\left(2\alpha''-\beta''+\frac{2\alpha'}{r}-\alpha'^2\right) = 8\pi\rho\frac{1-r\alpha'}{1-2r\alpha'} + 4\pi k^2 \exp(-2\beta). \tag{18}$$

Equations (16) to (18) are not, however, independent. The conservation relation of energy–stress and the Bianchi identity give an identical relation between them. We shall therefore take, as equations of our problem, (16) and the sum of (17) and (18):

$$\exp(2\alpha-2\beta)\left(\alpha''+\frac{\alpha'}{r}\right) = \frac{4\pi\rho}{1-2r\alpha'} + 4\pi k^2 \exp(-2\beta). \tag{19}$$

Thus we have two equations (16) and (19) involving three unknowns α, β and ρ. Thus we can choose α arbitrarily to construct a solution. As our purpose here is merely to exhibit the existence of a solution we make the simple choice

$$\alpha = \ln(k_1 + k_2 r^2/a^2) \tag{20}$$

where k_1 and k_2 are constants and a is the radius of the bounded distribution.

Now from equations (16) and (20) we obtain

$$\beta = 2\ln\left(k_1 + \frac{k_2 r^2}{a^2}\right) + 2\frac{k_1 - \pi k^2 a^2/k_2}{k_1 + k_2 r^2/a^2} + D \tag{21}$$

where D is a constant.

For the exterior static field we use the solutions given by Ghosh and Sengupta (1965)

$$\alpha = \ln\left(r^\lambda + \frac{\pi k^2}{(1-\lambda)^2} r^{2-\lambda}\right) \tag{22}$$

and

$$\beta = 2\alpha + \lambda(\lambda-2)\ln r + \ln B. \tag{23}$$

By continuity at $r = a$, we have

$$k_1 = \frac{1}{2}\left\{(2-\lambda)a^\lambda + \frac{\pi k^2}{(1-\lambda)^2}\lambda a^{2-\lambda}\right\} \tag{24}$$

$$k_2 = \frac{1}{2}\left\{\lambda a^\lambda + \frac{\pi k^2}{(1-\lambda)^2}(2-\lambda)a^{2-\lambda}\right\} \tag{25}$$

$$D = \lambda(\lambda-2)\ln a + \frac{\lambda(\lambda-2)[a^\lambda + \{\pi k^2/(1-\lambda)^2\}a^{2-\lambda}]}{[\lambda a^\lambda + \{\pi k^2/(1-\lambda)^2\}(2-\lambda)a^{2-\lambda}]} + \ln B. \tag{26}$$

A necessary condition for the metric to be regular at $r = 0$ is $\beta = 0$. This condition fixes the value of the constant B, which can now be evaluated from equations (21) and (26).

Thus equations (24) to (26) ensure the continuity of the metric tensor at the boundary. Further, we require the continuity of the normal component of the energy tensor. The continuity of the normal component of the energy tensor depends only on the continuity of E_1^1, the electromagnetic energy tensor. This is secured by the continuity of F_{13}, the only non-vanishing component of the electromagnetic field tensor and the continuity of the metric tensor. Now equation (11) gives, at $r = a$,

$$k_2 < \tfrac{1}{3}k_1. \tag{27}$$

Substituting values of k_1 and k_2 and after some simple calculation, we obtain

$$(1-\lambda)^2(\tfrac{1}{2}-\lambda) > \mu(\tfrac{3}{2}-\lambda) \tag{28}$$

where

$$\mu = \pi k^2 a^{2(1-\lambda)} \tag{29}$$

which is a positive quantity. Equation (28) can be satisfied only if either $\lambda < \frac{1}{2}$ or $\lambda > \frac{3}{2}$. For any value of $\lambda < \frac{1}{2}$, equation (28) sets an upper bound to μ, while for $\lambda > \frac{3}{2}$ equation (28) sets a lower bound to μ. In view of equation (29), both of these correspond to an upper bound for a, the radius of the matter distribution. Again, from equations (28), (20) and (21) we obtain

$$4\pi\rho = e^{-2\beta}\left(\frac{4k_1k_2}{a^2} - 4\pi k^2\right)(1 - 2r\alpha'). \tag{30}$$

Since matter density is positive definite, we have the condition from equation (30) that

$$k_1 k_2 > \pi k^2 a^2$$

i.e.

$$\lambda(2-\lambda)\left\{a^\lambda + \frac{\pi k^2}{(1-\lambda)^2}a^{2-\lambda}\right\} > 0$$

which shows that

$$0 < \lambda < 2. \tag{31}$$

Therefore from equation (28) we find that λ has two sets of values:

$$\begin{aligned} &\text{(i)} && 0 < \lambda < \tfrac{1}{2} \\ &\text{(ii)} && \tfrac{3}{2} < \lambda < 2. \end{aligned} \tag{32}$$

For either $\lambda = 0$ or $\lambda = 2$ our metric goes over to that of the Melvin universe. For $k = 0$ the solutions correspond to that obtained by the author previously for stationary cylindrically symmetric clusters of particles.

Solutions for two values of λ (λ_1 and λ_2) such that $\lambda = \lambda_1 + \lambda_2$ are essentially the same.

Acknowledgments

The author is deeply indebted to Professor A. K. Raychoudhuri of Presidency College, Calcutta, for his useful guidance.

References

EINSTEIN, A., 1939, *Ann. Math.*, **40,** 921.
GHOSH, R., and SENGUPTA, R., 1965, *Nuovo Cim.*, **38,** 1579.
MELVIN, M. A., 1964, *Phys. Lett.*, **8,** 65.
RAYCHOUDHURI, A. K., and SOM, M. M., 1962, *Proc. Camb. Phil. Soc.*, **58,** 338.
SOM, M. M., 1964, *Proc. Phys. Soc.*, **83,** 328.
—— 1967, *Proc. Phys. Soc.*, **90,** 1149.
SYNGE, J. L., 1960, *Relativity: The General Theory* (Amsterdam: North-Holland), p. 311.